THE INFLUENCE OF ANISOTROPY OF ELECTRICAL CONDUCTIVITY ON THE DEVELOPMENT OF FLOW OF A VISCOUS FLUID IN THE INITIAL SECTION OF A PLANE CHANNEL

(VLIIANIE ANIZOTROPII ELEKTROPROVODNOSTI NA RAZVITIE TECHENIIA VIAZKOI ZHIDKOSTI V NACHAL'NOM UCHASTKE PLOSKOGO KANALA)

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The development of the velocity profile in the flow of a viscous incompressible electrically conducting fluid in the initial section of channels of various shapes was first studied by approximate methods by Shercliff [1,2]. From the results there obtained it followed that as the intensity of the magnetic field increases the length of the inlet section decreases. Recently this problem in the case of plane and annular channels has been considered afresh by a number of workers [3-8].

On the other hand, the steady flow of a viscous incompressible medium in a plane channel of infinite length with allowance for anisotropy of electrical conductivity was investigated in [9,10].

We consider below the influence of anisotropy of conductivity on the development of flow in the initial section of a plane semi-infinite channel with nonconducting walls ($-a \leqslant y \leqslant a$, $x \geqslant 0$), in which there is a viscous electrically conducting medium having at the inlet section $x = 0$ a uniform velocity profile $u = u_0$. In the region of the channel $x \geqslant 0$ there is acting on the fluid an external uniform magnetic field B_0 parallel to the y-axis. The density ρ of the medium, the coefficient of viscosity η and the electroconductivity σ are assumed constant. If we introduce dimensionless variables

$$x = \frac{x^*}{a}, \quad y_1 = \frac{y^* \sqrt{R}}{a}, \quad z = \frac{z^*}{a}, \quad u = \frac{u^*}{u_0}, \quad v = \frac{v^* \sqrt{R}}{u_0}, \quad w = \frac{w^*}{u_0}$$

$$p = \frac{p^*}{\rho u_0^2}, \qquad \mathbf{B} = \frac{\mathbf{B}^*}{B_0}, \qquad \mathbf{E} = \frac{\mathbf{E}^*}{u_0 B_0}, \qquad \mathbf{j} = \frac{\mathbf{j}^*}{\sigma u_0 B_0} \tag{1}$$

(the dimensional quantities are marked with an asterisk) and seek a solution of the initial system (see equations (1) to (3) in [9]) in the form of series in powers of the small parameters R_m and $1/\sqrt{R}$, then for the zero order approximation we obtain

$$\begin{gathered}
\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} - Sj_z + \frac{\partial^2 u}{\partial y_1^2}, \qquad \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + Sj_x + \frac{\partial^2 w}{\partial y_1^2} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y_1} = 0, \qquad j_x - \omega\tau j_z = E_x - w, \qquad j_z + \omega\tau j_x = E_z + u \\
\frac{\partial p}{\partial y_1} = \frac{\partial E_x}{\partial y_1} = \frac{\partial E_z}{\partial y_1} = 0, \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0
\end{gathered} \tag{2}$$

where

$$S = \frac{\sigma B_0^2 a}{\rho u_0}, \qquad R = \frac{\rho u_0 a}{\eta}, \qquad R_m = \sigma\mu u_0 a$$

In writing down system (2) it has been assumed that the velocity of the medium does not depend on the z-coordinate, whilst the inertial terms, following Targ [11], are allowed for approximately. Similar simplifications can also be achieved by the method of estimates used by Liubimov in deriving the equations of the magnetohydrodynamic boundary layer [12].

Eliminating from the equations of motion the components of the current density and passing to the dimensionless variable $y = y_1\sqrt{R}$, we find that

$$\begin{gathered}
\frac{1}{R}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - \frac{S}{1+(\omega\tau)^2}(E_z - \omega\tau E_x + u + \omega\tau w) - \frac{\partial p}{\partial x} = 0 \\
\frac{1}{R}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial x} + \frac{S}{1+(\omega\tau)^2}(E_x + \omega\tau E_z + \omega\tau u - w) - \frac{\partial p}{\partial z} = 0
\end{gathered} \tag{3}$$

Differentiating (3) with respect to y and introducing the complex velocity $V = u + iw$, we obtain for V the following equation:

$$\frac{1}{R}\frac{\partial^3 V}{\partial y^3} - \frac{\partial^2 V}{\partial x \partial y} - \frac{S(1 - i\omega\tau)}{1+(\omega\tau)^2}\frac{\partial V}{\partial y} = 0 \tag{4}$$

In solving equation (4) we shall assume that the bulk flow in the direction of the x-axis is given, whilst the bulk flow in the direction of the z-axis is zero. Then we have the following boundary conditions for V:

$$V|_{x=0} = 1, \qquad V|_{y=\pm 1} = 0, \qquad Q = \int_{-1}^{+1} V dy = 2 \tag{5}$$

Applying to (4) and (5) the Laplace transformation and solving the equation so obtained, we find that

$$V^\circ = \frac{\cosh m - \cosh my}{\lambda(\cosh m - m^{-1}\sinh m)}, \qquad m = \sqrt{R\lambda + N^2(1 - i\omega\tau)} \tag{6}$$

where

$$V^\circ = \int_0^\infty V e^{-\lambda x}\, dx, \qquad N^2 = \frac{M^2}{1 + (\omega\tau)^2}, \qquad M^2 = SR = B_0^2 a^2 \frac{\sigma}{\eta}$$

Carrying out the inverse transformation, we obtain

$$V = V_\infty + 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{[\gamma_k^2 + N^2(1 - i\omega\tau)]\cos\gamma_k} \exp\left(-\frac{\gamma_k^2 + N^2(1 - i\omega\tau)}{R}x\right) \tag{7}$$

where

$$V_\infty = u_\infty + iw_\infty = \frac{\cosh N\sqrt{1 - i\omega\tau} - \cosh N\sqrt{1 - i\omega\tau}\, y}{\cosh N\sqrt{1 - i\omega\tau} - (N\sqrt{1 - i\omega\tau})^{-1}\sinh N\sqrt{1 - i\omega\tau}} =$$

$$= 1 - 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{[\gamma_k^2 + N^2(1 - i\omega\tau)]\cos\gamma_k}$$

and γ_k are the roots of the equation $\tan\gamma = \gamma$.

When $\omega\tau = 0$, formula (7) becomes the solution obtained by Shercliff

$$u = \frac{\cosh M - \cosh My}{\cosh M - M^{-1}\sinh M} + 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{(\gamma_k^2 + M^2)\cos\gamma_k} \exp\left(-\frac{\gamma_k^2 + M^2}{R}x\right), \qquad w = 0 \tag{8}$$

which when $M \to 0$ coincides with the results of Targ.

Separating the real and imaginary parts in (7), we find for u and w the following expressions:

$$u = u_\infty + 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{[(\gamma_k^2 + N^2)^2 + (\omega\tau N^2)^2]\cos\gamma_k} \Big[(\gamma_k^2 + N^2)\cos\frac{\omega\tau N^2}{R}x -$$

$$- \omega\tau N^2 \sin\frac{\omega\tau N^2}{R}x\Big] \exp\left(-\frac{\gamma_k^2 + N^2}{R}x\right) =$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{[(\gamma_k^2 + N^2)^2 + (\omega\tau N^2)^2]\cos\gamma_k} \Big\{\Big[(\gamma_k^2 + N^2)\cos\frac{\omega\tau N^2}{R}x -$$

$$- \omega\tau N^2 \sin\frac{\omega\tau N^2}{R}x\Big] \exp\left(-\frac{\gamma_k^2 + N^2}{R}x\right) - (\gamma_k^2 + N^2)\Big\} \tag{9}$$

$$w = w_\infty + 2\sum_{k=1}^{\infty} \frac{\cos\gamma_k y - \cos\gamma_k}{[(\gamma_k^2 + N^2)^2 + (\omega\tau N^2)^2]\cos\gamma_k} \Big[(\gamma_k^2 + N^2)\sin\frac{\omega\tau N^2}{R}x +$$

$$+ \omega\tau N^2 \cos \frac{\omega\tau N^2}{R} x \Big] \exp\left(-\frac{\gamma_k^2 + N^2}{R} x\right) =$$

$$= 2 \sum_{k=1}^{\infty} \frac{\cos \gamma_k y - \cos \gamma_k}{[(\gamma_k^2 + N^2)^2 + (\omega\tau N^2)^2] \cos \gamma_k} \left\{ \Big[(\gamma_k^2 + N^2) \sin \frac{\omega\tau N^2}{R} x + \right.$$

$$\left. + \omega\tau N^2 \cos \frac{\omega\tau N^2}{R} x \Big] \exp\left(-\frac{\gamma_k^2 + N^2}{R} x\right) - \omega\tau N^2 \right\} \quad (10)$$

From a comparison of formulas (8) with (9) and (10) it is clear that the anisotropy of conductivity leads to an increase in the length of the inlet section, since an increase in $\omega\tau$ leads to a decrease in the number N, which plays in the given case the role of the effective Hartmann number M.

To determine the pressure in the channel for known u and w we integrate equations (3) over the height of the channel. We obtain

$$\frac{\partial p}{\partial x} = \frac{1}{R}\left(\frac{\partial u}{\partial y}\right)_{y=1} - \frac{N^2}{R}(E_z + 1 - \omega\tau E_x), \quad \frac{\partial p}{\partial z} = \frac{1}{R}\left(\frac{\partial w}{\partial y}\right)_{y=1} + \frac{N^2}{R}(E_x + \omega\tau E_z + \omega\tau) \quad (11)$$

Differentiating the first equation with respect to z, and the second with respect to x, equating the mixed derivatives and making use of (2), we find for the components of the external electric field intensity E_x and E_z the system

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = -\frac{1}{N^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)_{y=1}, \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \quad (12)$$

which, with the boundary conditions $E_x = 0$, $E_z = -1$ when $x = 0$, has the solution

$$E_x = -\frac{1}{N^2}\left(\frac{\partial w}{\partial y}\right)_{y=1}, \qquad E_z = -1 \quad (13)$$

Then (11) takes the form

$$\frac{\partial p}{\partial x} = \frac{1}{R}\left(\frac{\partial u}{\partial y}\right)_{y=1} - \frac{\omega\tau}{R}\left(\frac{\partial w}{\partial y}\right)_{y=1}, \qquad \frac{\partial p}{\partial z} = 0 \quad (14)$$

and the pressure distribution in the channel is found by integrating (14) with the condition

$$p\,|_{x=0} = p_0$$

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